8.7.2021: Reformulation of the proposition and a corollary

The last proposition (saying under which conditions is $I_O(F,G) = mn + t$) can be reformulated into simpler version (with less words, but less context).

Proposition 0.1. Let F and G be curves defined by polynomials

$$F = F_m + F_{m+1} + \cdots, \tag{1}$$
$$G = G_n + G_{n+1} + \cdots, \tag{2}$$

such that F and G have t tangents in common at 0. Then

$$I_O(F,G) = mn + t, \tag{3}$$

(i.e. the correcting term $l = I_O(F,G) - mn - t$ is equal to zero) if an only if the one of the following conditions is satisfied for each common tangent L of F and G at O (of multiplicity r and s respectively):

- $r > s$ and F_{m+1} is not divisible by L.
- $r < s$ and G_{n+1} is not divisible by L.
- $r = s$ and exactly one of the polynomials F_{m+1}, G_{n+1} is divisible by L (and the other is not)
- $r = s$, both F_{m+1}, G_{n+1} are not divisible by L and $v_0 a_s \neq b_0 u_s$. In this case v_0, u_s, b_0, a_s $(a_s, u_s \neq 0)$ are the coefficients of F and G after the transformation which maps L onto y. After this transformation, the polynomials are

$$\begin{aligned} F &= [F_m] + [F_{m+1}] + \cdots = [a_s x^{m-s} y^s + \cdots + a_m y^m] + [b_0 x^{m+1} + \cdots + b_{m+1} y^{m+1}] + \cdots \\ G &= [G_n] + [G_{n+1}] + \cdots = [u_s x^{n-s} y^s + \cdots + u_n y^n] + [v_0 x^{n+1} + \cdots + v_{n+1} y^{n+1}] + \cdots \end{aligned} \tag{4}$$

I'm not sure if this is an improvement, it's almost the same. But anyway, the proof of the proposition above also implies the following.

Corollary. If the conditions above are not satisfied for a common tangent L, then the intersection multiplicity increases at least by the number of branches corresponding to this tangent. Concretely, we get

$$I_O(F,G) \geq mn + t + e \tag{5}$$

where

- if $r > s$, then e is the number of branches of F with the tangent L
- if $r < s$, then e is the number of branches of G with the tangent L
- if $r = s$, then e is the maximum of numbers of branches of F and G with the tangent L.

I believe this bound can be improved. There is a possibility this is obvious from blowups or something. I don't know yet.